Kinetic theory of geodesic acoustic and related modes

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Motivation

- Geodesic Acoustic Modes (GAM) are relatively high frequency eigen-modes supported by plasma compressibility in toroidal geometry.
- Coupled to the drift-wave and Zonal Flows?
- Coupled to Alfven modes/cascades? Can be driven by high energy partricles? Transport regulation/modulation? Coupling to high energy particles drive?
- GAM localization , radial propagation etc: dispersion properties.

Outline

- Extended MHD model to reproduce GAM kinetic dispersion relation
- Degeneration GAM (m = n = 0) and BAE (finite m and n)
- Coupling of GAM and BAE. Mode polarization.
- Dispersion of GAMS and BAE in various regimes: $\omega < k_0 v_{Te}$ or $\omega > k_0 v_{Te}$, $k_0 = (m - nq)/qR$
  - How does ideal MHD work in the limit of $\omega < k_0 v_{Te}$?
- Drift effects

Short history of GAM and related modes

•1968 : Geodesic acoustic modes: Winsor, Johnson, Dawson

•2000-05: Surge of interest related to zonal flows (GAM is an eigen mode of poloidal rotation in a tokamak)

- No theoretical or experimental work in between

- 2008: Every large or small tokamak has seen one or several variety of GAM; many sightings in numerical simulations

-But: Mysterious ubiquitous 25 kHz mode on many tokamaks, 1970-2000

1973: Mikhailovskii, NF: Electromagnetic drift wave instabilities

(finite m,n GAM/BAE)

1977: Mazur, Mikhailovskii, NF, Beam driven Alfven waves: 7/4 coefficient surfaces

1999: Mikhailovskii, Sharapov, : Electromagnetic drift wave instabilities, Plasma Phys Reports, GAM+BAE+ drift effects

1996: Levedev, Yushmanov, Diamond, Smolyakov, PoP,: Relaxation of poloidal rotation problem, 7/4 surfaces again from kinetic calculations

Short history of GAM cont'd

- 1993, Heidbrink et al, "What is the beta-induced Alfven eigen-mode?" oscillations with $\omega \approx v_{Ti} / R$
- 1992: Chu, Green et al., Coupling of Alfven and sound continuum via geodesic curvature creates low frequency gap
- 1996-2008: Zonca et al., Unstable Alfven modes in the continuous spectrum: GAM dispersion relation with 7/4, electromagnetic (Alfven modes) effects but no references to Winsor, Green Johnson; AITG modes
- 2001--2008: Berk, Sharapov, Gorelenkov, Fu, Nazikian, and others: Alfven cascades, BAE/Alfven waves zoology, BAAE (Gorelenkov), Fu (EGAM), …

Discrepancy between MHD and kinetic theory

MHD theory GAM mode polarization includes: $\tilde{\phi}^{(0)}(r,t)$, $\tilde{p}^{(1)}$ and $\tilde{V}_{\parallel}^{(1)}$) (finite q coupling) to the longitudinal sound wave

$$\omega^2 = \frac{c_s^2}{R^2}\left(2 + \frac{1}{q^2}\right),$$

where $c_s^2 = \gamma p_0/\rho_0$. The first term is the GAM part, the $1/q^2$ term is due to the sound coupling.

Kinetic theory gives

$$\omega^2 = \frac{v_{Ti}^2}{R^2}\left(\frac{7}{4} + \frac{1}{\tau}\right).$$

MHD $\rightarrow$ kinetic: $2\gamma = 10/3 \rightarrow 7/2 + 1/\tau$. Geodesic compressibility index 7/4, is a result of different compressibility of parallel and perpendicular pressure Smolyakov, 2005.

Extended MHD for GAMs

Plasma quasineutrality $\nabla_\perp \cdot \mathbf{J}_\perp + \nabla_\| J = 0$ with

$$\mathbf{J}_\perp = \frac{1}{\omega_{ci}} \mathrm{b} \times \frac{d\mathbf{V}_E}{dt} + \frac{c}{B} \mathrm{b} \times \boldsymbol{\nabla} p + \frac{c}{B} \mathrm{b} \times \boldsymbol{\nabla} \cdot \boldsymbol{\Pi}.$$

The perpendicular current contains the contribution of parallel viscosity $\boldsymbol{\Pi}{=}3\pi_\| \left(bb{-}I/3\right)/2$, which is related to the pressure anisotropy, $\pi_\| = 2(p_\| - p_\perp)/3$. results in

$$-\frac{d}{dt} \frac{e n_0 c}{B_0 \omega_{ci}} \nabla_\perp^2 \phi - \frac{2c}{B_0} \frac{\partial}{\partial r} \left(p + \frac{\pi_\|}{4} \right) \frac{\partial}{r \partial \theta} \ln B + \nabla_\| J = 0,$$

The evolution of the viscosity tensor is governed by the Grad type equation Mikhailovskii 1984, Smolyakov 1998.

$$\frac{d\boldsymbol{\Pi}}{dt} + \boldsymbol{\Pi} \nabla \cdot \mathbf{V} + \left[\boldsymbol{\Pi} \cdot \nabla \mathbf{V} + (\boldsymbol{\Pi} \cdot \nabla \mathbf{V})^T - \frac{2}{3} \mathbf{I} \, (\boldsymbol{\Pi} : \nabla \mathbf{V}) \right]$$
$$+ \omega_c (\mathrm{b} \times \boldsymbol{\Pi} - \boldsymbol{\Pi} \times \mathrm{b}) + \left[p \nabla \mathbf{V} + p (\nabla \mathbf{V})^T - \frac{2}{3} \mathbf{I} \; p \nabla \cdot \mathbf{V} \right]$$
$$+ \frac{2}{5} \left[\nabla \mathbf{q} + (\nabla \mathbf{q})^T - \frac{2}{3} \mathbf{I} \; \nabla \cdot \mathbf{q} \right] + \nabla \cdot \tau = 0.$$

Mode polarization and coupling

All perturbed quantities

$$X = X_0 + \left(\widehat{X}_1 \exp(i\theta) + \widehat{X}_{-1} \exp(-i\theta)\right) \exp(-im\theta + in\zeta) + ...$$

Here $X_0 \sim \exp(-im\theta + in\zeta)$ is the principal component , and $\widehat{X}_1$ and $\widehat{X}_{-1}$ are side-bands due to the geodesic curvature

$$k_0 = \frac{m - nq}{qR}, \quad k_{\pm 1} = \frac{m \pm 1 - nq}{qR}$$

$$m \simeq nq, \qquad k_{\pm 1} \simeq \pm \frac{1}{qR} \gg k_0$$

It is convenient to work with $\widehat{X}_1 \pm \widehat{X}_{-1}$ combinations:

$$\widehat{X}_{-1} \exp(i\theta) + \widehat{X}_{-1} \exp(i\theta) = \left(\widehat{X}_1 + \widehat{X}_{-1}\right) \cos\theta + i\left(\widehat{X}_1 - \widehat{X}_{-1}\right) \sin\theta = \widehat{X}_c$$

Extended MHD (no electrons) GAM with m=n=0

$$\frac{d}{dt}p \; - \frac{10}{3}p_{0i}\mathbf{V}_E \cdot \nabla \ln B = 0.$$

$$\frac{d}{dt}\pi_\parallel \; - \frac{2}{3}p_{0i}\mathbf{V}_E \cdot \nabla \ln B = 0.$$

Coupling of potential, plasma pressure and viscosity

$$-i\frac{\omega e n_0 c k_r^2}{B_0\omega_{ci}}\phi_0 + \frac{ck_r}{B_0 R}\;\left(\widehat{p}_1 - \widehat{p}_{-1} + \frac{\widehat{\pi}_1 - \widehat{\pi}_{-1}}{4}\right)\; = 0,$$

$$-i\omega\left(\widehat{p}_1 - \widehat{p}_{-1} + \frac{\widehat{\pi}_1 - \widehat{\pi}_{-1}}{4}\right) + \frac{7}{4}\frac{p_0 c k_r}{B_0 R}\left(\widehat{\phi}_2 - \widehat{\phi}_{-2}\right) - \frac{7}{2}\frac{p_0 c k_r}{B_0 R}\phi_0 = 0.$$

These equations reproduce kinetic dispersion relation (7/4) , Mazur, Mikhailovskii 1977,Lebedev 1996, Zonca 1996.

$$\omega^2 = \frac{7}{4}\frac{v_{Ti}^2}{R^2} = \frac{7}{2}\frac{p_{0i}}{\rho}\frac{1}{R^2},$$

For $m = n = 0$ principal harmonic, the mode is mostly electro-static.

Finite m and m GAM with electromagnetic effects

For finite m and n GAM couples to the principal Alfvén branch in the main order and the dispersion relation is

$$\omega^2 = \frac{7}{4}\frac{v_{Ti}^2}{R^2} + k_0^2 v_A^2,$$

Mikhailovskii 1973, Zonca 1996, Breizman 2005, $k_0 \equiv k_{\|0} = (m - nq)/qR$.

Mode polarization (in the main order, without dispersion):

$\left(\widehat{p}_1 - \widehat{p}_{-1} + \left(\widehat{\pi}_1 - \widehat{\pi}_{-1}\right)/4\right)$, ϕ_0 and A_0

The first electromagnetic mode

This mode involves coupled , $p_0 + \pi_0/4$, $\left(\widehat{\phi}_1 - \widehat{\phi}_{-1}\right)$, $\left(\widehat{A}_1 + \widehat{A}_{-1}\right)$, $\widehat{p}_2 + \widehat{p}_{-2}$, and $\widehat{\pi}_2 + \widehat{\pi}_{-2}$

$$-i\frac{\omega e n_0 c k_r^2}{B_0 \omega_{ci}}\left(\widehat{\phi}_1 - \widehat{\phi}_{-1}\right) - \frac{2ck_r}{B_0 R}\left(p_0 + \frac{\pi_0}{4}\right) + \frac{ck_r}{B_0 R}\left(\widehat{p}_2 + \widehat{p}_{-2} + \frac{\widehat{\pi}_2 + \widehat{\pi}_{-2}}{4}\right)$$

$$+ \frac{i}{qR}\left(\widehat{J}_1 + \widehat{J}_{-1}\right) = 0.$$

$$-i\omega\left(p_0 + \frac{\pi_0}{4}\right) + \frac{7}{4}\frac{p_0 c k_r}{B_0 R}\left(\widehat{\phi}_1 - \widehat{\phi}_{-1}\right) = 0,$$

$$-i\omega\left(\widehat{p}_2 + \widehat{p}_{-2} + \frac{\widehat{\pi}_2 + \widehat{\pi}_{-2}}{4}\right) - \frac{7}{4}\frac{p_0 c k_r}{B_0 R}\left(\widehat{\phi}_1 - \widehat{\phi}_{-1}\right) = 0,$$

Dispersion relation

$$\omega^2 = \frac{21}{8}\frac{v_{Ti}^2}{R^2} + \frac{v_A^2}{q^2 R^2}.$$

Alfven mode shifted by average geodesic curvature (in fact, this is an MHD result, $v_{Ti}^2 = 2p_0/\rho$)

The second electromagnetic mode

This mode involves $\left(\widehat{\phi}_1+\widehat{\phi}_{-1}\right)$, $\left(\widehat{A}_1-\widehat{A}_{-1}\right)$, $\widehat{p}_2-\widehat{p}_{-2}$, and $\widehat{\pi}_2-\widehat{\pi}_{-2}$

$$-i\frac{\omega e n_0 c k_r^2}{B_0\omega_{ci}}\left(\widehat{\phi}_1+\widehat{\phi}_{-1}\right)+\frac{ck_r}{B_0R}\left(\widehat{p}_2-\widehat{p}_{-2}+\frac{\widehat{\pi}_2-\widehat{\pi}_{-2}}{4}\right)+\frac{i}{qR}\left(\widehat{J}_1-\widehat{J}_{-1}\right)=$$

$$-i\omega\left(\widehat{p}_2-\widehat{p}_{-2}+\frac{\widehat{\pi}_2-\widehat{\pi}_{-2}}{4}\right)-\frac{7}{4}\frac{p_0ck_r}{B_0R}\left(\widehat{\phi}_1+\widehat{\phi}_{-1}\right)=0,$$

Dispersion relation

$$\omega^2=\frac{7}{8}\frac{v_{Ti}^2}{R^2}+\frac{v_A^2}{q^2R^2}.$$

Another Alfven mode shifted by average geodesic curvature. Note two different values of the geodesic shift $\frac{7}{8}\frac{v_{Ti}^2}{R^2}$ and $\frac{21}{8}\frac{v_{Ti}^2}{R^2}$

Three modes:

GAM /BAE

$$\omega^2 = \frac{7}{4}\frac{v_{Ti}^2}{R^2} + k_0^2 v_A^2,$$

$$\phi_0,\ A_0,\ \widehat{p}^{(1)} \sim \sin\theta$$

Two electromagnetic modes: split Alfven waves

$$\omega^2 = \frac{7}{8}\frac{v_{Ti}^2}{R^2} + \frac{v_A^2}{q^2 R^2}.$$

$$\widehat{\phi}^{(1)} \sim \sin\theta,\ \widehat{A}^{(1)} \sim \cos\theta,\ p_0,\ \text{and}\ \widehat{p}^{(2)} \sim \cos 2\theta$$

$$\omega^2 = \frac{21}{8}\frac{v_{Ti}^2}{R^2} + \frac{v_A^2}{q^2 R^2}.$$

$$\widehat{\phi}^{(1)} \sim \cos\theta,\ \widehat{A}^{(1)} \sim \sin\theta,\ \text{and}\ \widehat{p}^{(2)} \sim \sin 2\theta$$

Coupling to sound continuum in kinetic theory/extended MHD theory:

Small corrections of the order of $v_{Ti}/(\omega q R) << 1$

For of $m = n = 0$,

$$\omega^2 = \frac{7}{4}\frac{v_{Ti}^2}{R^2}\left(1 + \frac{46}{49}q^{-2}\right).$$

The factor $1 + 46/\left(49q^2\right)$ replaces the coefficient $1 + 1/\left(2q^2\right)$ (Sugama 2006,Zonca 2007,Gao 2006,Zonca 2008)

For finite m and n, the kinetic calculations lead to

$$\omega^2 = \frac{7}{4}\frac{v_{Ti}^2}{R^2}\left(1 + \frac{23}{14}\frac{v_{Ti}^2}{q^2R^2\omega^2}\right) + k_0^2 v_A^2$$

Coupling to acoustic continuum become more important for Alfvén side-band modes.

$$\omega^2 = \frac{21}{8}\frac{v_{Ti}^2}{R^2}\left(1 + \frac{69}{14}\frac{v_{Ti}^2}{q^2R^2\omega^2}\right) + \frac{v_A^2}{q^2R^2}$$

$$\omega^2 = \frac{7}{8}\frac{v_{Ti}^2}{R^2}\left(1 + \frac{23}{2}\frac{v_{Ti}^2}{q^2R^2\omega^2}\right) + \frac{v_A^2}{q^2R^2}.$$

Interaction of Alfven and acoustic continua

Gorelenkov 2007, Holst 2000 Standard MHD model

$$i\left(\omega^2\nabla_\perp^2\phi + v_A^2\nabla_\parallel\nabla_\perp^2\nabla_\parallel\phi\right) - \frac{\omega B_0 k_r}{n_0 m_i c R}p\left(e^{i\theta} - e^{-i\theta}\right) = 0,$$

$$i\left(\omega^2 + c_s^2\nabla_\parallel^2\right)p - \frac{\gamma c p_0 \omega k_r}{B_0}\phi\left(e^{i\theta} - e^{-i\theta}\right) = 0,$$

where $c_s^2 = \gamma p_0/\rho_0 = \gamma v_{Ti}^2/2$, and $\rho_0 = n_0 m_i$.

Finite m and n GAM and BAAE gap

Coupling of the Alfvén continuum at the principal and acoustic side-bands harmonics

$$-\left(\omega^2 - k_0^2 v_A^2\right) + \frac{2c_s^2}{R^2}\,\frac{\omega^2}{\omega^2 - \quad c_s^2/q^2R^2} = 0$$

This mode primarily involves ϕ_0, A_0, and $\widehat{p}_1 - \widehat{p}_{-1}$.
For small $k_0^2 v_A^2$ there are two modes here .
GAM mode:

$$\omega^2 \simeq \frac{2c_s^2}{R^2}\left(1 + c_s^2/\omega^2 q^2 R^2\right) + k_0^2 v_A^2 .or$$

$$\omega^2 \simeq \frac{2c_s^2}{R^2}\left(1 + \frac{1}{2q^2}\right) + k_0^2 v_A^2.$$

And the second lower frequency mode:

$$\omega^2 \simeq \frac{k_0^2 v_A^2}{2q^2}.$$

These two modes form a BAAE gap (Gorelenkov 2007). The third mode appears when shear is included, $k_1 \neq -k_{-1}$

Interaction of Alfven and acoustic continua

The first electromagnetic side-band mode

Coupled oscillations of the side-band Alfvén, acoustic principal harmonic p_0, and second order acoustic side-bands

$$-\left(\omega^2-\frac{v_A^2}{q^2R^2}\right)+\frac{2c_s^2}{R^2}\frac{\omega^2}{\omega^2-k_0^2c_s^2}+\frac{c_s^2}{R^2}\frac{\omega^2}{\omega^2-4c_s^2/q^2R^2}=0.$$

The second electromagnetic side-band mode

The second electromagnetic branch involves Alfvén side-band and second order acoustic side-bands

$$-\left(\omega^2-\frac{v_A^2}{q^2R^2}\right)+\frac{c_s^2}{R^2}\frac{\omega^2}{\omega^2-4c_s^2/q^2R^2}=0.$$

Dispersion effects on GAMs/BAE

$$k_{\|0} = (m - nq)/qR \ll k_{\|}^{\pm 1} \simeq 1/qR$$

Electron response to side-band fluctuations is always adiabatic:

$$\omega < k_{\|}^{\pm 1}\ v_{Te}$$

For a principal component with $k_{\|0}$, there are two possible regimes:

adiabatic, $\omega \ll k_0 v_{Te}$ and hydrodynamic regime (e.g. $m = n = 0$), $\omega \gg k_0 v_{Te}$

Electron response calculated from drift kinetic equation in two regimes $\omega \ll k_0 v_{Te}$ and $\omega \ll k_0 v_{Te}$ taking into account the next order corrections of the order of $\omega_{de}^2/\omega^2 \ll 1$

The ion response is calculated in the fluid limit $\omega \gg v_{Ti}/qR$, $\omega \gg \omega_{di}$ up to the second order in $\omega_{di}^2/\omega^2 \ll 1$ and including the polarization effect, $k_r^2 \rho_i^2 \ll 1$. Quasineutrality $n_i = n_e$ and Ampere law are used get the dispersion equations.

Ion kinetic response

$$\left(\omega - \omega_D - k_\parallel v_\parallel\right) f = (\omega\ \ )\frac{eF_0}{T_i}\left(\phi - \frac{v_\parallel}{c}A\right) J_0^2(k_\perp v_\perp/\omega_{ci}) - \left(\omega - \omega_D - k_\parallel v_\parallel\right) \frac{e}{}$$

fluid limit $\omega > \widehat{\omega}_d$ and expansion in $\widehat{\omega}_d/\omega$.

$$\widetilde{f}_0 = \frac{e\phi_0}{T_i} F_m \left[J_0^2(k_\perp v_\perp/\omega_{ci}) - 1 + \frac{1}{2}\frac{\omega_d^2}{\omega^2} J_0^2(k_\perp v_\perp/\omega_{ci}) \ + \frac{3\omega_d^4}{8\omega^4} J_0^2(k_\perp v_\perp/\omega_{ci})\right.$$

$$+\frac{i}{2}\frac{\omega_d}{\omega} J_0^2(k_\perp v_\perp/\omega_{ci}) \left[1 + \frac{3\omega_d^2}{4\omega^2}\right] \frac{eF_M}{T_i}\left(\phi_{-1} - \phi_{+1}\right)$$

$$\widehat{f}^{(1)} = \ \frac{i}{2}\frac{\omega_d}{\omega} J_0^2(k_\perp v_\perp/\omega_{ci}) \left[1 + \frac{3}{4}\frac{\omega_d^2}{\omega^2}\right] \frac{eF_M}{T_i}\phi_0 \left(e^{i\theta} - e^{-i\theta}\right)$$

$$+\left[J_0^2(k_\perp v_\perp/\omega_{ci}) - 1 + \frac{1}{2}\frac{\omega_d^2}{\omega^2} J_0^2(k_\perp v_\perp/\omega_{ci}) \ \right] \frac{eF_m}{T_i}\left(\phi_1 e^{i\theta} + \phi_{-1} e^{-i\theta}\right)$$

$$-\frac{1}{4}\frac{\omega_d^2}{\omega^2} J_0^2(k_\perp v_\perp/\omega_{ci}) \ \frac{eF_M}{T_i}\left(\phi_1 e^{-i\theta} + \phi_{-1} e^{i\theta}\right).$$

Electron drift-kinetic equation

The electron dynamics is described by the drift kinetic equation

$$\left(\omega-\omega_D-k_{\|}v_{\|}\right)g=-\omega\left(\phi-\frac{v_{\|}}{c}A\right)\ \frac{eF_m}{T_e},$$

Separating the principal and oscillating components in one obtains

$$\omega g_0-\langle\widehat{\omega}_d\widehat{g}\rangle\ -k_0v_{\|}g_0=-\omega\left(\phi_0-\frac{v_{\|}}{c}A_0\right)\ \frac{eF_m}{T_e},$$

$$\omega\widehat{g}-\widehat{\omega}_dg_0-\left(\widehat{\omega}_d\widehat{g}-\langle\widehat{\omega}_d\widehat{g}\rangle\right)-\left(k_0+\widehat{k}\right)v_{\|}\widehat{g}\ =-\omega\left(\widehat{\phi}-\frac{v_{\|}}{c}\widehat{A}\right)\ \frac{eF_m}{T_e},$$

where $< ... >$ means the average in θ.

$$\widehat{g}_1+\widehat{g}_{-1}=-\frac{\omega\left[\left(\omega-k_0v_{\|}\right)^2-\overline{\omega}_{de}^2/2\right]}{\left(\omega-k_0v_{\|}\right)\Delta}\left(\widehat{\phi}_1+\widehat{\phi}_{-1}-\frac{v_{\|}}{c}\left(\widehat{A}_1+\widehat{A}_{-1}\right)\right)\ \frac{e}{T_e}F_m$$

$$-\frac{v_{\|}\omega}{\Delta qR}\left(\widehat{\phi}_1-\widehat{\phi}_{-1}-\frac{v_{\|}}{c}\left(\widehat{A}_1-\widehat{A}_{-1}\right)\right)\ \frac{e}{T_e}F_m+i\frac{\omega\overline{\omega}_{de}v_{\|}}{\left(\omega-k_0v_{\|}\right)\Delta qR}\ \left(\phi_0-\frac{v_{\|}}{c}A_0\right.$$

Electron drift kinetic equation

$$\widehat{g}_1 - \widehat{g}_{-1} = -\frac{\omega\left(\omega - k_0 v_\|\right)}{\Delta}\left(\widehat{\phi}_1 - \widehat{\phi}_{-1} - \frac{v_\|}{c}\left(\widehat{A}_1 - \widehat{A}_{-1}\right)\right)\ \frac{e}{T_e}F_m$$

$$-\frac{v_\|\omega}{qR\Delta}\left(\widehat{\phi}_1 + \widehat{\phi}_{-1} - \frac{v_\|}{c}\left(\widehat{A}_1 + \widehat{A}_{-1}\right)\right)\ \frac{e}{T_e}F_m + i\frac{\omega\overline{\omega}_{de}}{\Delta}\left(\phi_0 - \frac{v_\|}{c}A_0\right)\ \frac{eF_m}{T_e},$$

$$g_0 = -\frac{i}{2}\frac{\overline{\omega}_{de}\omega}{\Delta}\left(\widehat{\phi}_1 - \widehat{\phi}_{-1} - \frac{v_\|}{c}\left(\widehat{A}_1 - \widehat{A}_{-1}\right)\right)\ \frac{e}{T_e}F_m$$

$$-\frac{i}{2}\frac{v_\|\overline{\omega}_{de}\omega}{qR\left(\omega - k_0 v_\|\right)\Delta}\left(\widehat{\phi}_1 + \widehat{\phi}_{-1} - \frac{v_\|}{c}\left(\widehat{A}_1 + \widehat{A}_{-1}\right)\right)\ \frac{e}{T_e}F_m$$

$$-\frac{\omega\left[\left(\omega - k_0 v_\|\right)^2 - v_\|^2/q^2R^2\right]}{\left(\omega - k_0 v_\|\right)\Delta}\left(\phi_0 - \frac{v_\|}{c}A_0\right)\ \frac{e}{T_e}F_m,$$

$$\Delta = \left(\omega - k_0 v_\|\right)^2 - v_\|^2/q^2R^2 + \overline{\omega}_{de}^2/2.$$

The first Alfven side band in the regime of adiabatic electrons: $\omega < k_0 v_{Te}$

Note that $\omega < k_0 v_{Te}$ does not allow for the case $k_0 \to 0$. Side-band Alfvén oscillations with $\left(\widehat{\phi}_1 + \widehat{\phi}_{-1}\right) \sim \cos\theta$ and $\left(\widehat{A}_1 - \widehat{A}_{-1}\right) \sim \sin\theta$ parity

$$\tau \left[\left(\widehat{\phi}_1 + \widehat{\phi}_{-1}\right) - \frac{\omega q R}{c} \left(\widehat{A}_1 - \widehat{A}_{-1}\right)\right] = \left[\Gamma_0 - 1 + \frac{1}{4} K_2\right]\left(\phi_1 + \phi_{-1}\right),$$

$$\left[k_r^2 \lambda_{De}^2 - \frac{\omega^2 q^2 R^2}{c^2}\right]\left(\widehat{A}_1 - \widehat{A}_{-1}\right) = -\frac{\omega q R}{c}\left(\widehat{\phi}_1 + \widehat{\phi}_{-1}\right) \quad .$$

where $\tau = T_i / T_e$.

$$\omega^2 = \frac{v_A^2}{q^2 R^2} + \frac{7}{8}\frac{v_{Ti}^2}{R^2} + \frac{v_A^2}{q^2 R^2} k_r^2 \rho_s^2 \left(1 - \frac{7}{8}\frac{v_{Ti}^2}{\omega^2 R^2}\right).$$

Dispersion due to finite ion-sound Larmor radius, $\rho_s^2 = T_e / \left(m_i \omega_{ci}^2\right)$.

GAM and the second Alfven side band, $\omega < k_0 v_{Te}$: ϕ_0 and $\left(\widehat{\phi}_1 - \widehat{\phi}_{-1}\right) \sim \sin\theta$

For finite m and n the principal A_0 and side-band $\left(\widehat{A}_1 + \widehat{A}_{-1}\right) \sim \cos\theta$ components of the magnetic vector potential become important.

$$\left[\Gamma_0 - 1 + \frac{1}{2}K_{2i}\right]\phi_0 - \frac{i}{2}K_{1i}\left(\phi_1 - \phi_{-1}\right) = \tau\left(\phi_0 - \frac{\omega}{k_0 c}A_0\right),$$

$$iK_{1i}\phi_0 + \left[\Gamma_0 - 1 + \frac{3}{4}K_{2i}\right]\left(\phi_1 - \phi_{-1}\right) = \tau\left[\left(\widehat{\phi}_1 - \widehat{\phi}_{-1}\right) - \frac{\omega q R}{c}\left(\widehat{A}_1 + \widehat{A}_{-1}\right)\right]$$

$$\left(k_r^2\lambda_{De}^2 - \frac{\omega^2}{k_0^2 c^2}\right)A_0 + \frac{i}{2}\frac{\overline{\omega}_{de}\omega q R}{k_0 c^2}\left(\widehat{A}_1 + \widehat{A}_{-1}\right) = -\frac{\omega}{k_0 c}\phi_0,$$

and

$$i\frac{\overline{\omega}_{de}\omega q R}{k_0 c^2}A_0 + \left(\frac{\omega^2 q^2 R^2}{c^2} - k_r^2\lambda_{De}^2\right)\left(\widehat{A}_1 + \widehat{A}_{-1}\right) = \frac{\omega q R}{c}\left(\widehat{\phi}_1 - \widehat{\phi}_{-1}\right).$$

Two modes with $\left(\widehat{\phi}_1 - \widehat{\phi}_{-1}\right) \sim \sin\theta$ and $\left(\widehat{A}_1 + \widehat{A}_{-1}\right) \sim \cos\theta$:

The lower frequency GAM mode

$$\omega^2 = \omega_1^2 \equiv \frac{v_{Ti}^2}{R^2}\left(\frac{7}{4} + \frac{1}{\tau}\right) + k_0^2 v_A^2,$$

and the higher frequency side-band Alfvén mode

$$\omega^2 = \omega_2^2 \equiv \frac{v_{Ti}^2}{R^2}\left(\frac{21}{8} + \frac{1}{\tau}\right) + \frac{v_A^2}{q^2 R^2}.$$

Dispersion of the GAM mode

$$\omega^2 = \omega_1^2 + \frac{k_0^4 v_A^4}{\omega_1^2} k_r^2 \rho_s^2 - \frac{1}{2}\frac{\overline{\omega}_{de}^2}{\omega_1^2} k_0^2 v_A^2 + \frac{\overline{\omega}_{de}^2}{2\omega_1^2}\frac{v_{Ti}^2}{R^2}\left(\frac{7}{8} - \frac{1}{\tau}\right).$$

There are two types of the dispersive effects here: ion-sound Larmor radius effects, $k_r^2 \rho_s^2 < 1$; and average geodesic curvature, $\omega_{de}^2/\omega^2 < 1$. Dispersion of Alfvén side band

$$\omega^2 = \omega_2^2 + \frac{v_A^2}{q^2 R^2} k_r^2 \rho_s^2,$$

this is well known dispersion of Alfvén waves.

Regime of hydrodynamic electrons $\omega > k_0 v_{Te}$:

The first Alfvén side-band oscillations: $\left(\phi_1 + \phi_{-1}\right)$ and $\left(\hat{A}_1 - \hat{A}_{-1}\right)$

The same as for the adiabatic regime

$$\omega^2 = \frac{v_A^2}{q^2 R^2} + \frac{7}{8}\frac{v_{Ti}^2}{R^2} + \frac{v_A^2}{q^2 R^2} k_r^2 \rho_s^2 \left(1 - \frac{7}{8}\frac{v_{Ti}^2}{\omega^2 R^2}\right).$$

GAM + the second Alfven side band in the hydrodynamic regime $(\omega > k_0 v_{Te})$:

Coupled perturbations of $\left(\phi_1 - \phi_{-1}\right)$, $\left(\widehat{A}_1 + \widehat{A}_{-1}\right)$, A_0 and ϕ_0

$$\left[\Gamma_0 - 1 + \frac{1}{2}K_{2i}\right]\phi_0 - \frac{i}{2}K_{1i}\left(\phi_1 - \phi_{-1}\right) = -\frac{k_0^2 v_{Te}^2 \tau}{2\omega^2}\left(\phi_0 - \frac{\omega}{k_0 c}A_0\right) - \frac{i}{2}\frac{\tau\overline{\omega}_{de}}{c}$$

$$iK_{1i}\phi_0 + \left[\Gamma_0 - 1 + \frac{3}{4}K_{2i}\right]\left(\phi_1 - \phi_{-1}\right) = \tau\left[\left(\widehat{\phi}_1 - \widehat{\phi}_{-1}\right) - \frac{\omega qR}{c}\left(\widehat{A}_1 + \widehat{A}_{-1}\right)\right.$$

$$k_r^2\lambda_{De}^2\left(\widehat{A}_1 + \widehat{A}_{-1}\right) = -\frac{\omega qR}{c}\left[\widehat{\phi}_1 - \widehat{\phi}_{-1} - \frac{\omega qR}{c}\left(\widehat{A}_1 + \widehat{A}_{-1}\right)\right] + i\frac{\overline{\omega}_{de}qR}{c}\phi_0.$$

$$\left(\frac{k_r^2 c^2}{\omega_{pe}^2} + 1\right)A_0 = \frac{k_0 c}{\omega}\phi_0 - i\frac{\overline{\omega}_{de}qRc}{v_{Te}^2}\left[\widehat{\phi}_1 + \widehat{\phi}_{-1} - \frac{\omega qR}{c}\left(\widehat{A}_1 - \widehat{A}_{-1}\right)\right],$$

GAM + the second Alfven side band in the hydrodynamic regime ($\omega > k_0 v_{Te}$) :

GAM mode

$$\omega^2 = \omega_1^2 \equiv \frac{v_{Ti}^2}{R^2}\left(\frac{7}{4} + \frac{1}{\tau}\right) + k_0^2 v_A^2,$$

The side-band Alfvén mode

$$\omega^2 = \frac{v_A^2}{q^2 R^2} + \frac{21}{8}\frac{v_{Ti}^2}{R^2}.$$

almost the same as for the adiabatic regime (but no extra $1/\tau$ term)

GAMs in the hydrodynamic $\omega > k_0 v_{Te}$ and adiabatic $\omega < k_0 v_{Te}$ regimes are the same

$$\omega^2 = \omega_1^2 \equiv \frac{v_{Ti}^2}{R^2}\left(\frac{7}{4} + \frac{1}{\tau}\right) + k_0^2 v_A^2,$$

Dispersion is different

Ideal MHD in adiabatic and hydrodynamic regime for electrons

Hydrodynamic regime, $\omega > k_0 v_{Te}$:

$$\frac{\partial}{\partial t} p + \mathbf{V}_E \cdot \nabla p = 0$$

and adiabatic regime, $\omega < k_0 v_{Te}$:

$$\nabla_{\|} p = 0$$

give the same response if $\mathbf{E}_{\|} = 0$. In other words,

$$\frac{\partial}{\partial t} \tilde{p} + \mathbf{V}_E \cdot \nabla p_0 = 0$$

and

$$\frac{\tilde{B}_{\perp}}{B_0} \tilde{p} + \frac{\mathbf{B}_0}{B_0} \cdot \nabla p_0 = 0$$

are equivalent if

$$-\nabla_{\|} \phi - \frac{1}{c} \frac{\partial A}{\partial t} = 0$$

Dispersion of GAM modes

GAM mode

$$\omega^2 = \omega_1^2 \equiv \frac{v_{Ti}^2}{R^2}\left(\frac{7}{4}+\frac{1}{\tau}\right) + k_0^2 v_A^2,$$

$$\omega^2 = \omega_1^2 - \frac{k_r^2 c^2}{\omega_{pe}^2}\frac{k_0^2 v_A^2}{\omega^2} - \frac{1}{2}\frac{\overline{\omega}_{de}^2}{\omega_1^2}k_0^2 v_A^2 + \frac{\overline{\omega}_{de}^2}{2\omega_1^2}\frac{v_{Ti}^2}{R^2}\left(\frac{7}{8}-\frac{1}{\tau}\right).$$

In this regime, $\omega > k_0 v_{Te}$, the dispersion correction are due to a finite electron inertia, $k_r^2 c^2/\omega_{pe}^2 < 1$, and a finite magnetic drift frequency, $\omega_{de}^2/\omega^2 < 1$. Sign of the ω_{de}^2/ω^2 dispersion is the same as in the adiabatic regime Sign of the $k_r^2 c^2/\omega_{pe}^2$ term is opposite to the sign of the $k_r^2 \rho_s^2$ dispersion in the adiabatic regime:

$$\omega^2 = \omega_1^2 + \frac{k_0^4 v_A^4}{\omega_1^2}k_r^2 \rho_s^2 - \frac{1}{2}\frac{\overline{\omega}_{de}^2}{\omega_1^2}k_0^2 v_A^2 + \frac{\overline{\omega}_{de}^2}{2\omega_1^2}\frac{v_{Ti}^2}{R^2}\left(\frac{7}{8}-\frac{1}{\tau}\right).$$

Consistent with the dispersion of Alfvén waves in a slab plasma.

Drift effects on GAMs with high m

or

Dispersion and instability of drift waves due to the average geodesic curvature

Ion drift-kinetic equation

$$f = -\frac{e}{T_i} F_{m0} \phi + g,$$

$$\left(\omega - \omega_d - k_\parallel v_\parallel\right) g = \left(\omega - \widehat{\omega}_*\right) \frac{e}{T_i} \left(\phi - \frac{v_\parallel}{c} A\right) J_0^2 (k_\perp v_\perp / \omega_{ci}) F_0.$$

$$\widehat{\omega}_* = \omega_{*i} \left(1 + \eta_i \left(\frac{v^2}{v_{th}^2} - \frac{3}{2}\right)\right),$$

drift frequency $\omega_{*i} = k_y T_i n_0' / (e B_0 n_0)$, $k_y = m/r$; magnetic drift frequency $\omega_d = \frac{v_\perp^2/2 + v_\parallel^2}{\omega_{ci}} \mathbf{k} \cdot \mathrm{b} \times \nabla \ln B = \frac{i}{2} \widehat{\omega}_{di} \left(\exp(i\theta) - \exp(-i\theta)\right).$

Principal and side-band harmonics:

$$X = X_m \exp(-im\theta + in\zeta) + X_{m\pm1} \exp(-i(m \pm 1)\theta + in\zeta).$$

Fluid expansions, no resonances

Fluid regime (both for the main and side-band components), no Landau damping, no electromagnetic effects

$$\omega > \left(k^{\|}_{m,n} v_{Ti}, k^{\|}_{m\pm 1,n} v_{Ti},\right), \qquad \omega > \omega_D$$

$$\begin{aligned} \tilde{f}_m &= -\frac{e\phi_m}{T_i} F_m + \left(1 - \frac{\widehat{\omega}_*}{\omega}\right) J_0^2(k_\perp v_\perp/\omega_{ci}) \frac{e\phi_m}{T_i} F_0 \left(1 + \frac{\widehat{\omega}_{di}^2}{2\omega^2}\right) \\ &+ \frac{i}{2}\left(1 - \frac{\widehat{\omega}_*}{\omega}\right) J_0^2(k_\perp v_\perp/\omega_{ci}) \frac{e\ F_m}{T_i} \frac{\widehat{\omega}_{di}}{\omega} \left(\phi_{m-1,n} - \phi_{m+1,n}\right), \end{aligned}$$

$$\begin{aligned} \tilde{f}_{m\pm 1,n} &= -\frac{e\phi_{m\pm 1,n}}{T_i} F_m + \left(1 - \frac{\widehat{\omega}_*}{\omega}\right) J_0^2(k_\perp v_\perp/\omega_{ci}) \frac{e\phi_{m\pm 1,n}}{T_i} F_m \\ &\pm \frac{i}{2}\left(1 - \frac{\widehat{\omega}_*}{\omega}\right) J_0^2(k_\perp v_\perp/\omega_{ci}) \frac{e\ F_m}{T_i} \frac{\widehat{\omega}_{di}}{\omega} \phi_{m,n}. \end{aligned}$$

Assume that $m > 1$ and neglect the difference between $\omega_{*i}^{m\pm 1}$ and ω_{*i}^m, $\omega_*^{m\pm 1} \simeq \omega_*^m = \omega_{*i}$.

Ion density response

$$\tilde{n}_{m,n} = -\frac{\omega_{*i}}{\omega}\frac{e\phi_{m,n}}{T_i}n_0 - \frac{k_\perp^2 v_T^2}{2\omega_{ci}^2}\left(1 - \frac{\omega_{*i}(1+\eta_i)}{\omega}\right)\frac{e\phi_{m,}}{T_i}n_0$$

$$+\frac{7}{8}\frac{\overline{\omega}_{di}^2}{\omega^2}\left(1 - \frac{\omega_{*i}(1+2\eta)}{\omega}\right)\frac{e\phi_{m,n}}{T_i} + \frac{i}{2}\frac{\overline{\omega}_{di}}{\omega}\left(1 - \frac{\omega_{*i}(1+\eta)}{\omega}\right)\frac{en_0}{T_i}\left(\phi_{m-1,n} - \phi_{m+1,n}\right)$$

and

$$\tilde{n}_{m\pm 1} = -\frac{\omega_{*i}}{\omega}\frac{e\phi_{m\pm 1}}{T_i}n_0 \pm \frac{i}{2}\frac{\overline{\omega}_{di}}{\omega}\left(1 - \frac{\omega_{*i}(1+\eta)}{\omega}\right)\frac{e\ \phi_m}{T_i}n_0\ .$$

Electrons are adiabatic

$$\tilde{n}_e = \frac{e\phi_{m\pm 1}}{T_e}n_0$$

Dispersion and instability of drift waves

Electron drift waves

Ion sound and FLR dispersion

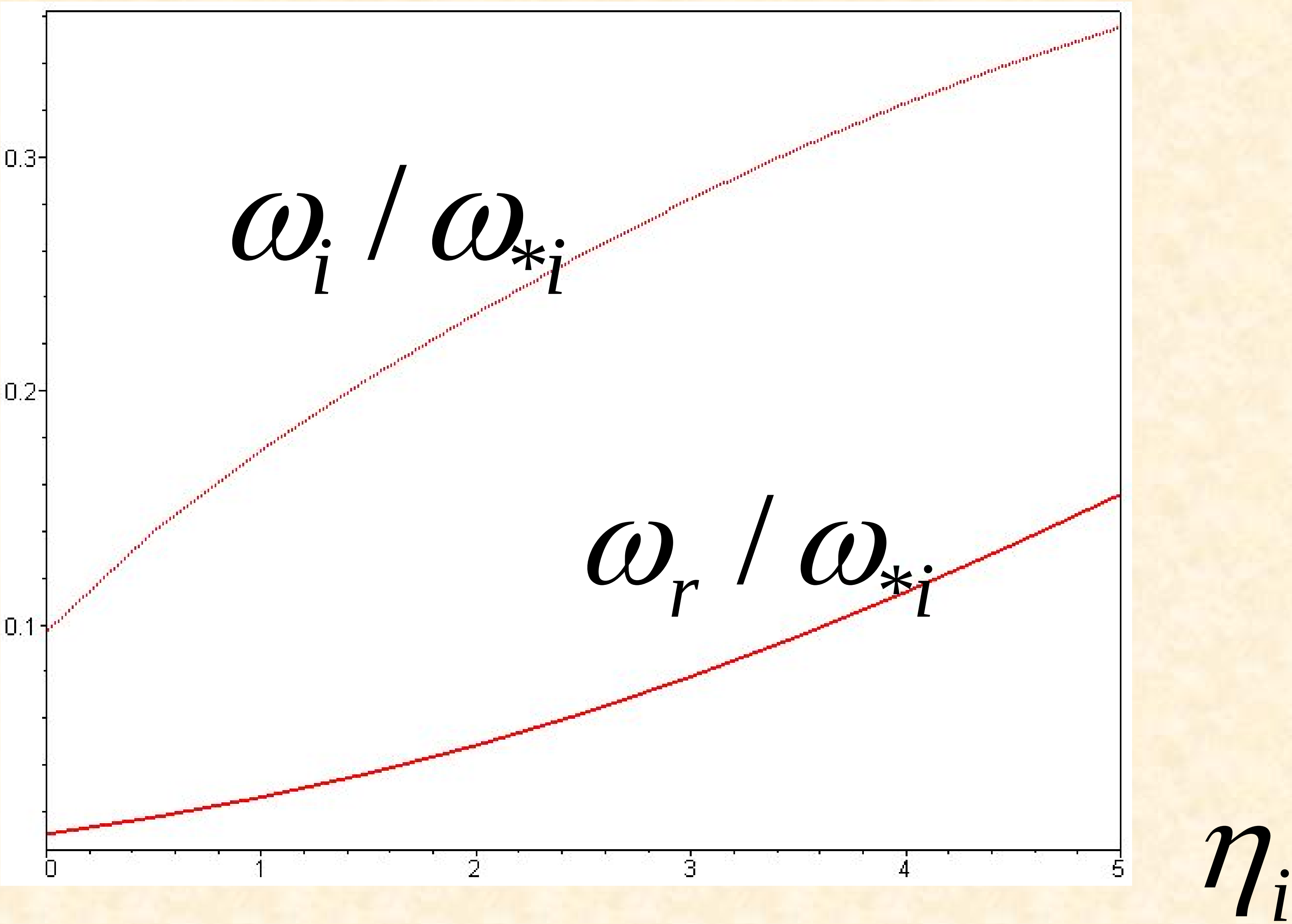
$$-\tau - \frac{\omega_{*i}}{\omega} - \left(1 - \frac{\omega_{*i}(1+\eta_i)}{\omega}\right)\frac{k_\perp^2 v_T^2}{2\omega_{ci}^2} + \frac{7}{8}\frac{\overline{\omega}_{di}^2}{\omega^2}\left(1 - \frac{\omega_{*i}(1+2\eta_i)}{\omega}\right)$$

$$+ \frac{1}{2}\frac{\overline{\omega}_{di}^2}{\omega^2}\left(1 - \frac{\omega_{*i}(1+\eta_i)}{\omega}\right)^2 \frac{1}{\tau + \omega_{*i}/\omega} = 0.$$

Average geodesic curvature dispersion

Generalized inertia

Ion drift mode destabilized by averaged geodesic curvature



Summary

Three related electromagnetic eigen-modes that involve average geodesic curvature: –Two of these, involve a typical GAM parity with perturbations of ϕ_0, A_0 and $\left(\widehat{\phi}_1 - \widehat{\phi}_{-1}\right), : \left(\widehat{A}_1 + \widehat{A}_{-1}\right)$. –One of these is GAM with finite m and n (BAE). – The second mode of the same parity is the Alfvén side-band –The third eigen-mode involves $\left(\widehat{\phi}_1 + \widehat{\phi}_{-1}\right) \sim \cos\theta$ and $\left(\widehat{A}_1 - \widehat{A}_{-1}\right) \sim \sin\theta$ perturbations. This mode is essentially electromagnetic and represents another Alfvén side-band shifted by the average geodesic curvature

Direct coupling of GAM and Alfvén side-bands, may provide new important channel affecting drift-wave turbulence in a tokamak. The geodesic curvature shift of Alfven mode can create the conditions for mode localization (eigen-modes)

Summary cont'd

Dispersion of GAM:

-Finite electron magnetic drift frequency, $\overline{\omega}_{de}^2/\omega^2$.

-Finite ion-sound Larmor radius, $k_r^2\rho_s^2$, in the adiabatic regime $\omega < k_0 v_{Te}$, and finite electron inertia $k_r^2 c^2/\omega_{pe}^2$ in the hydrodynamic regime.

-Finite ion magnetic drift frequency, ω_{di}^2/ω^2 (neglected here, Zonca 1996, 2008)

$k_r^2\rho_s^2$ and $k_r^2 c^2/\omega_{pe}^2$ dispersion have the opposite signs (similar to the slab plasma case).

-Average geodesic curvature provides a generalized inertia term with a sign opposite to that of the ion polarization/FLR. In combination with drift effects potentially may be destabilizing.